Double Integration.

In this chapter, we shall discuss the evaluation of

$$\int_a^b \int_c^d f(x,y)\,dx\,dy \quad \text{using} \quad \begin{array}{l} \text{(i) Trapezoidal rule} \\ \text{(ii) Simpson's rule.} \end{array}$$

<u>Note:</u> Weighting factors at the points on the boundary and interior points

① 3×3 ~~order~~ matrix

Trapezoidal rule.

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 1 \end{bmatrix}$$

Simpson's rule

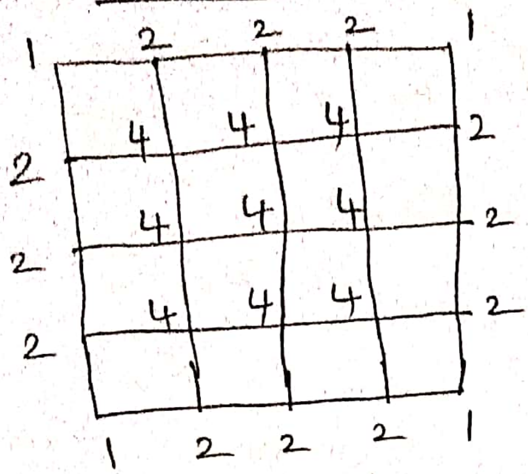
$$\begin{bmatrix} 1 & 4 & 1 \\ 4 & 16 & 4 \\ 1 & 4 & 1 \end{bmatrix}$$

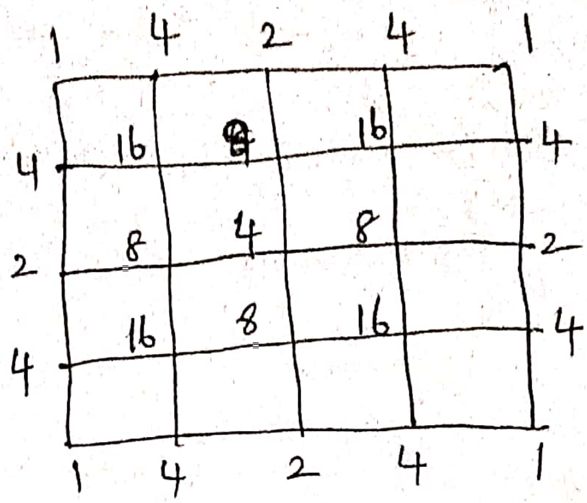
② 4×4 matrix

$$\begin{bmatrix} 1 & 2 & 2 & 1 \\ 2 & 4 & 4 & 2 \\ 2 & 4 & 4 & 2 \\ 1 & 2 & 2 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 4 & 2 & 1 \\ 4 & 16 & 8 & 4 \\ 2 & 8 & 4 & 2 \\ 1 & 4 & 2 & 1 \end{bmatrix}$$

3 5 x 5 matrix





Formula.

1) Trapezoidal rule.

$$I = \frac{hk}{4}$$ [(sum of values of f at the four corners)
+ 2 (sum of the values of f at the remaining nodes on the boundary) + 4 (sum of the values of f at the interior nodes)]

2) Simpson's rule

$$I = \frac{hk}{9}$$ [(sum of the values of f at the four corners) +
2 (sum of the values of f at the odd position on the boundary except the corners) +
4 (sum of the values of f at the even position on the boundary) +
4 (sum of the values of f at odd position on the odd row of the matrix except boundary rows) +
8 (sum of the values of f at even position on the odd row of the matrix except boundary rows) +
8 (sum of the values of f at the odd positions on even rows) +
16 (sum of the values of f at the even positions on the even rows of the matrix)]

Note!

x axis → h direction & y axis → k direction.

Problems.

1. Evaluate the integral $\int_1^2\int_1^2 \frac{dx\,dy}{x+y}$. Using trapezoidal rule with $h=k=0.25$

Solution:

G-T $f(x,y) = \frac{1}{x+y}$ & $h = 0.25$
$k = 0.25$

y ↓ \ x →	1	1.25	1.5	1.75	2
1	0.5 (1)	0.444 (2)	0.4 (2)	0.3636 (2)	0.3333 (1)
1.25	0.444 (2)	0.4 (4)	0.3636 (4)	0.3333 (4)	0.3077 (2)
1.5	0.4 (2)	0.3636 (4)	0.3333 (4)	0.3077 (4)	0.2857 (2)
1.75	0.3636 (2)	0.3333 (4)	0.3077 (4)	0.2857 (4)	0.2667 (2)
2	(1) 0.3333	(2) 0.3077	(2) 0.2857	(2) 0.2667	(1) 0.25

$I = \frac{hk}{4}\Big[$ (sum of values of f at the four corners) + 2 (Sum of the values of f at the remaining nodes on the boundary) + 4 (sum of the values of f at the interior nodes) $\Big]$

$$= \frac{(0.25)(0.25)}{4}\Big[(0.5+0.333+0.3333+0.25) + 2(0.444+0.4+0.3636+0.3077+0.2857+0.2667+0.3333+0.3077+0.2857+0.444+0.4+0.3636) + 4(0.4+0.3636+0.3333+0.3636+0.3333+0.3077+0.3333+0.3077+0.2857)\Big]$$

$I = 0.34065$

$\boxed{I \simeq 0.3407}$

2) Evaluate $\int_1^{1.5}\int_1^{2} \frac{dx\,dy}{x+y}$ using simpson's rule with $h = 0.5$ and $k = 0.25$

Solution:

G.T $I = \int_1^{1.5}\int_1^{2} \frac{dx\,dy}{x+y}$ & $h = 0.5$, $k = 0.25$

y↓ \ x→	1	1.5	2
1	0.5 (1)	0.4 (4)	0.3333 (1)
1.25	0.444 (4)	0.3636 (16)	0.3077 (4)
1.5	0.4 (1)	0.3333 (4)	0.2829 (1)

$I = \frac{hk}{9}\Big[$ (Sum of the values of f at the four corners) + 4(Sum of the values of f at the ~~[illegible]~~ on the boundary ~~[illegible]~~ except corners values) + ~~except boundary [illegible]~~ 16(Sum of the interior values)

$= \frac{(0.5)(0.5)}{9}\Big[(0.5 + 0.3333 + 0.2829 + 0.4) + 4(0.4 + 0.3077 + 0.444 + 0.3333) + 16(0.3636)\Big]$

$\boxed{I = 0.1844}$

(3) Evaluate $\int_0^{\pi/2}\int_0^{\pi/2} \sin(x+y)\,dx\,dy$ by using

(i) Trapezoidal rule, (ii) Simpson's rule, taking $h = \frac{\pi}{4}$ & $k = \frac{\pi}{4}$.

<u>Solution:</u>

G.T $I = \int_0^{\pi/2}\int_0^{\pi/2} \sin(x+y)\,dx\,dy$

$h = \frac{\pi}{4}$ & $k = \frac{\pi}{4}$

( ) → Trapezoidal

[ ] → Simpson's rule. (weight factor)

$y \downarrow$ \ $x \rightarrow$	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$
0	0 (1) [1]	0.7071 (2) [4]	1 (1) [1]
$\frac{\pi}{4}$	0.7071 (1) [1]	1	0.7071 (2) [4] (1) [1]
$\frac{\pi}{2}$	1	(2) [4] 0.7071	0

By Trapezoidal rule.

$I = \frac{hk}{4}$ [(Sum of four corner values of f) + 2(Sum of the values of f at the remaining nodes on the boundary) + 4(Sum of the values of f at the interior values)]

$$= \frac{\frac{\pi}{4} \times \frac{\pi}{4}}{4}\left[(0+1+1+0)+2(0.7071+0.7071+0.7071+0.7071) + 4(1)\right]$$

$$\boxed{I = 1.7975}$$

By Simpson's rule

$$I = \frac{hk}{9}\Big[(\text{Sum of the values of } f \text{ on the cornervalues}) + 4(\text{sum of the values of } f \text{ at the remaining nodes on the boundary}) + 16(\text{Sum of the values of } f \text{ at the interior values})\Big]$$

$$= \frac{\frac{\pi}{4}\times\frac{\pi}{4}}{9}\Big[(0+1+1+0)+4(0.7071+0.7071+0.7071+0.7071) + 16(1)\Big]$$

$$\boxed{I = 2.0080}$$

4) Evaluate $\displaystyle\int_0^1\int_1^2 \frac{2xy}{(1+x^2)(1+y^2)}\,dy\,dx$

(i) Trapezoidal rule (ii) Simpson's rule,

taking $h = k = 0.25$

Solution:

Given that $I = \displaystyle\int_0^1\int_1^2 \frac{2xy}{(1+x^2)(1+y^2)}\,dy\,dx$

where $f(x,y) = \dfrac{2xy}{(1+x^2)(1+y^2)}$

$h = 0.25$ & $k = 0.25$

$x \downarrow$ \ $y \rightarrow$	1	1.25	1.5	1.75	2
0	0 (1) [1]	0 (2) [4]	0 (2) [2]	0 (2) [4]	0 (1) [1]
0.25	0.2353 (2) [4]	0.2296 (4) [16]	0.2172 (4) [8]	0.2027 (4) [16]	0.1649 (2) [4]
0.5	0.4 (2) [2]	0.3902 (4) [8]	0.3692 (4) [4]	0.3446 (4) [8]	0.32 (2) [2]
0.75	0.48 (2) [4]	0.4683 (4) [16]	0.4431 (4) [8]	0.4135 (4) [16]	0.3840 (2) [4]
1	0.5 (1) [1]	0.4878 (2) [4]	0.4615 (2) [2]	0.4308 (2) [4]	0.4 (1) [1]

( ) → Trapezoidal weightfactor

[ ] → Simpson's weightfactor

<u>By Trapezoidal rule.</u>

$$I = \frac{hk}{4}\Big[\text{(sum of values of } f \text{ at the four corners)} + 2(\text{sum of the values of } f \text{ at the remaining nodes on the boundary}) + 4(\text{sum of the values of } f \text{ at the interior values})\Big]$$

$$= \frac{(0.25)(0.25)}{4}\Big[(0+0+0.5+0.4) + 2(0.2353+0.4+0.48+0.4878+0.4615+0.4308+0.1649+0.32+0.3840+0+0+0) + 4(0.2296+0.2172+0.2027+0.3902+0.3692+0.3446+0.4683+0.4431+0.4135)\Big]$$

$$\boxed{I = 0.3116.}$$

By simpson rule.

$$I = \frac{hk}{9}\Big[$$ (sum of the values of f at the four corners) +
2 (sum of the values of f at the odd position on the boundary except the corner) +
4 (sum of the values of f at the even position on the boundary) +
4 (sum of the values of f at odd position on the odd row) +
8 (sum of the values of f at even position on the odd row) +
8 (sum of the values of f at the odd position on the even row) +
16 (sum of the values of f at the even position on the even row)$\Big]$

$$= \frac{(0.25)(0.25)}{9}\Big[(0+0+0.5+0.4) + 2\underbrace{(0+0.32+0.4615+0.4)}_{\text{odd position}} + 4\underbrace{(0+0+0.16495+0.3840+0.4878+0.4308+0.2353+0.48)}_{\text{even position}} + 4\underbrace{(0.3692)}_{\text{odd row odd position}} +$$

$$8\underbrace{(0.3902+0.3446)}_{\text{even position on odd row}} + 8\underbrace{(0.2172+0.4431)}_{\text{odd position on even row}} + 16\underbrace{(0.2296+0.2027+0.4683+0.4135)}_{\text{even position on even row.}}\Big]$$

$$\boxed{I = 0.3171}$$

Home work.

① Evaluate $\int_0^1 \int_0^1 e^{x+y} \, dy\,dx$, using Trapezoidal rule and simpson's rule, with $h=k=0.25$

Ans. Trapezoidal rule $I = 3.0763$

Simpson's rule $I = 2.9545$

② Evaluate $\int_1^2 \int_3^4 \frac{dx\,dy}{(x+y)^2}$, taking $h=k=0.5$ using both Trapezoidal rule & simpson's rule.

Ans : Trapezoidal rule $I = 0.0413$

Simpson's rule $I = 0.0408$

③ Evaluate $\int_0^1 \int_0^1 (x^2+y^2) \, dx\,dy$ using Trapezoidal rule with $h=k=0.25$.

Ans $I = 0.6875$.